JOURNAL OF APPLIED PHYSICS VOLUME 27, NUMBER 1 JANUARY, 1956

Concavity of Resistance Functions

C. E. SHANNON AND D. W. HAGELBARGER
Bell Telephone Laboratories, Inc., Murray Hill, New Jersey

(Received August 1, 1955)

It is proved that any network of linearly wound potentiometers and fixed resistors has a curve of resistance *versus* shaft angle which is concave downward.

As part of a computer, a rheostat having a resistance that was a concave upward function of the shaft angle was needed. After many attempts to approximate it with networks of linearly wound potentiometers and fixed resistors, it became apparent that either it was impossible or that we were singularly inept network designers. Rather than accept the latter alternative, we have proved the following theorem:

Theorem: Let N be a two-terminal network of resistors, $R_1, R_2, \cdots, R_n$ with resistance R between its two terminals. Then $R=R(R_1, R_2, \cdots, R_n)$ is a concave downward function of $R_1, R_2 \cdots, R_n$.

This means that for any two sets of (non-negative) values $R_1, R_2, \cdots, R_n$ and $R_1', R_2', \cdots, R_n'$ we will have:

$$R\left(\frac{R_1+R_1'}{2}, \frac{R_2+R_2'}{2}, \cdots, \frac{R_n+R_n'}{2}\right)$$

$$\geq \tfrac{1}{2}[R(R_1, R_2, \cdots R_n)+R(R_1', R_2', \cdots, R_n')]. \quad (1)$$

Thinking of these sets of resistances as two points in n-space, this states that the resistance R at the midpoint of the joining line is greater than or equal to the average of the values at the ends.

Before proving the theorem we state two corollaries.

Corollary I: Let $R_{ij}=(\partial^2 R/\partial R_i \partial R_j)$. Then R_{ij} is a negative semidefinite form.

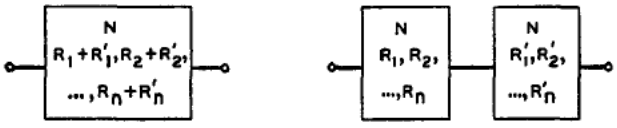

FIG. 1.

Corollary II: Any network of linearly wound potentiometers[1] on a common shaft and fixed resistors will give a resistance which is a concave function of shaft position. This is also true if potentiometers are used which are wound to be concave functions of shaft position.

All three terminals of a potentiometer may be used in the network, since this is equivalent to two rheostats with their wipers connected.

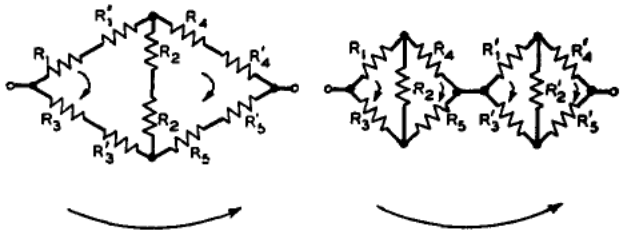

FIG. 2.

Proof: The right-hand side of Eq. (1) may be interpreted as half the resistance of a series connection of the network N with $R_1, R_2, \cdots, R_n$ for its elements and the network N with $R_1', R_2', \cdots, R_n'$ for elements. The left-hand term is the resistance of the network N with elements $(R_1+R_1')/2$, $(R_2+R_2')/2$, $\cdots$, $(R_n+R_n')/2$ or (due to linearity) half the resistance with elements R_1+R_1', R_2+R_2', $\cdots$, R_n+R_n'. Thus, to establish Eq. (1), we must show that in Fig. 1 the box at the left has the same or higher resistance than the series connection at the right. A simple special case of these networks is shown in Fig. 2.

[1] A linearly wound potentiometer is one whose resistance is a linear function of shaft position.

To the network with elements R_1+R_1', R_2+R_2', $\cdots$, R_n+R_n' add the auxiliary circuits shown in Fig. 3 for the particular network of Fig. 2. The additions consist of a set of ideal one-to-one transformers connected across the unprimed elements. A set of independent meshes is chosen for the network N. Suppose elements $a, b, \cdots, k$ form one of these meshes. Then the other winding of transformers connected across elements $R_a, R_b, \cdots, R_k$ are connected in series. This is done for each mesh in the set chosen except the mesh containing the external source. If these series connections are now left open it may be seen that the ideal transformers have no effect whatever and the circuit acts like the left-hand circuit of Fig. 1 or, in the particular case, Fig. 2. On the other hand, if the loops described for these ideal transformers are closed (switches SW_1 and SW_2 in Fig. 3 are closed), the constraints of the circuit are exactly the constraints of the right-hand part of Fig. 1 or, in the particular case, Fig. 2. It may be seen that the ideal transformers imply the mesh constraints for the unprimed elements. The original mesh constraints minus those due to the ideal transformers imply that the primed variables also satisfy these constraints. The mesh through the external source contains the same elements in both cases. Consequently, the current through each element and the voltage across it in Fig. 3 is the same as the corresponding current and voltage in Fig. 2 right, and the resistance of Fig. 3 when these switches are closed is the same as Fig. 2 right.

In any linear resistance network, closing a switch reduces (or leaves constant) the resistance. This is

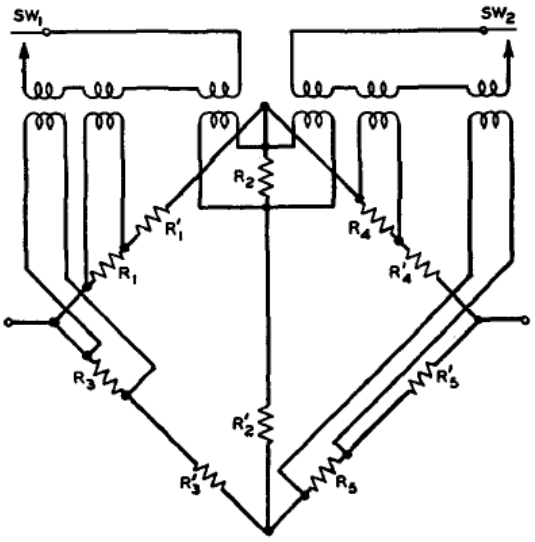

FIG. 3.

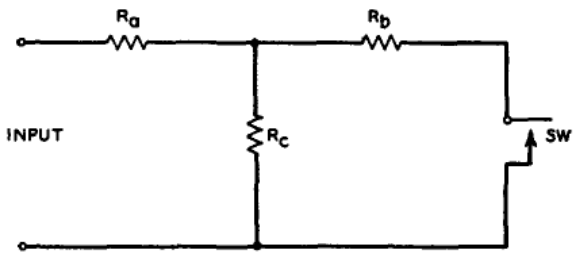

FIG. 4.

more or less evident from physical intuition but follows mathematically by representing the two terminal circuit and switch by its equivalent T structure, Fig. 4. If the original circuit is passive then although the individual resistances R_a, R_b, and R_c may be negative, R_b+R_c is non-negative. The input resistance with the switch open is R_a+R_c and with it closed is $R_a+R_bR_c/(R_b+R_c)$. Now since

$$R_c^2 \geq 0$$
$$R_bR_c+R_c^2 \geq R_bR_c$$
$$R_c \geq R_bR_c/(R_b+R_c)$$
$$R_a+R_c \geq R_a+R_bR_c/(R_b+R_c).$$

Thus closing the switch reduces the resistance. In Fig. 3 this applies to each switch and hence the resistance of the right part of Fig. 2 is less than or equal to that of the left-hand side. This argument, of course, applies to the general networks of Fig. 1 and proves our main theorem.

The ideal transformers in this argument are not really necessary—the argument could be translated into purely mathematical terms. A set of equations is found of the mesh type which can be transformed into those for either of the two circuits in Fig. 1 by making a set of particular resistors (the switches) have zero or infinite resistance. The standard network theorems work for these mesh equations, in particular the result concerning reduction of resistance when a switch is closed. The ideal transformers merely serve as a convenient way of describing how to set up these equations.

Corollary I is a well-known condition equivalent to convexity of a function when these second partials exist.[2] These partials do exist for physical networks, since the resistance R is then a bilinear function of any particular R without poles at finite points.

The second corollary is equivalent to the statement that a convex function in n-space reduces to a convex function of one variable if we move uniformly along a straight line in the n-space.

[2] Hardy, Littlewood, and Polya, *Inequalities* (Cambridge University Press, London, 1934), p. 80.